Problem Set 9 – Statistical Physics B

Problem 1: Infinite-range Ising model

Consider a one-dimensional Ising model consisting of N spins. Each spin interacts with *every* other spin with an exchange coupling $-J/N$ with $J > 0$. The energy of a spin configuration $\{s_i\}$ is given by

$$E(\{s_i\}) = -\frac{J}{2N}\sum_{i,j} s_i s_j,$$

where each spin variable can take the values $s_i = \pm 1$ with $i = 1, ..., N$. The canonical partition function is

$$Z(N,T) = \sum_{\{s_i\}} \exp\left(\frac{J}{2Nk_\mathrm{B}T}\sum_{i,j} s_i s_j\right).$$

(a) Perform a Hubbard-Stratonovich transformation to the variable m, which satisfies $\langle m \rangle = \langle N^{-1}\sum_i s_i\rangle$. Show that the partition function is given by

$$Z(N,T) = \sqrt{\frac{NJ}{2\pi k_\mathrm{B}T}}\int_{-\infty}^{\infty} dm\, \exp\left[-\frac{Nf_\mathrm{L}(m)}{k_\mathrm{B}T}\right],$$

where

$$f_\mathrm{L}(m) = \frac{J}{2}m^2 - k_\mathrm{B}T\ln\left[2\cosh\left(\frac{mJ}{k_\mathrm{B}T}\right)\right].$$

(b) For $T < T_\mathrm{c}$ the system exhibits a second order phase transition with m as order parameter. What are the physical differences between the $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$ case? Explain your answer in detail.

(c) Determine the critical temperature T_c.

(d) For $T \uparrow T_\mathrm{c}$, we find $\langle m \rangle \propto (T_\mathrm{c} - T)^\beta$. Determine the critical exponent β.

(e) We now add a magnetic field, such that

$$f_\mathrm{L}(m,B) = \frac{J}{2}m^2 - k_\mathrm{B}T\ln\left[2\cosh\left(\frac{mJ}{k_\mathrm{B}T}\right)\right] - mB.$$

We define the free energy per spin as

$$F(T,B) = -k_\mathrm{B}T\lim_{N\to\infty}\frac{1}{N}\ln\left\{\int_{-\infty}^{\infty} dm\, \exp\left[-\frac{Nf_\mathrm{L}(m)}{k_\mathrm{B}T}\right\}\right].$$

Show that $F(T,B)$ is non-analytic for $T < T_\mathrm{c}$. In particular, show that $F(T < T_\mathrm{c}, B) \propto |B|^\eta$ and determine η.

(f) The infinite-range Ising model predicts a phase transition in 1D, whereas there is no phase transition in the 1D Ising model with just nearest-neighbour interactions. Explain this.